International Journal of Modern Physics A,

ASYMPTOTICALLY ANTI-DE SITTER SPACETIMES AND THEIR STRESS ENERGY TENSOR

KOSTAS SKENDERIS*
Physics Department, Princeton University
Princeton, NJ 08544, USA

We consider asymtotically anti-de Sitter spacetimes in general dimensions. We review the origin of infrared divergences in the on-shell gravitational action, and the construction of the renormalized on-shell action by the addition of boundary counterterms. In odd dimensions, the renormalized on-shell action is not invariant under bulk diffeomorphisms that yield conformal transformations in the boundary (holographic Weyl anomaly). We obtain formulae for the gravitational stress energy tensor, defined as the metric variation of the renormalized on-shell action, in terms of coefficients in the asymptotic expansion of the metric near infinity. The stress energy tensor transforms anomalously under bulk diffeomorphisms broken by infrared divergences.

1. Introduction

Anti-de Sitter (AdS) spacetimes were studied in the eighties because they appear as ground states of many gauged supergravities. In recent times they have attracted a lot of attention due to the AdS/CFT correspondence. The construction of conserved charges for asymptotically AdS spaces was addressed in the eighties, see for instance Refs.[1]. Recent work appeared in Ref.[2]. The AdS/CFT duality has provided us with new insights and results about AdS gravity. It is the purpose of this contribution to extract these results and put them in a purely gravitational context.

Our considerations will be at the classical level. We would however like to view our results as the lowest order approximation to the quantum theory. The (logarithm of the) partition function of gravity is given to lowest order by the on-shell value of the gravitational action. It is thus essential that the on-shell action is finite. For asymptotically AdS solutions the on-shell action is proportional to the volume of spacetime and thus diverges since the spacetime is non-compact. One may deal with this divergence by regulating the on-shell action, computing all infinities, adding counterterms to cancel them and then removing the regulator. The resulting renormalized on-shell action is manifestly finite. This procedure was carried out in Ref. [3].

The renormalized on-shell action does not preserve all symmetries that the unrenormalized (and thus infinite) on-shell action formally does. This is so because some of the infrared divergences do not preserve all symmetries. Adding counterterms to the action to cancel the infinities results in a renormalized on-shell action

* kostas@feynman.princeton.edu

that transforms anomalously under some of the original symmetries. In particular, it was found in Ref.[3] that in odd dimensions the counterterms break the bulk diffeomorphisms that generate Weyl transformations on the boundary. This means that the renormalized on-shell action depends on the chosen metric on the boundary, not just on its conformal class. This anomaly corresponds, via the AdS/CFT correspondence, to the conformal anomaly of the dual CFT. We would like to emphasize that the anomaly does not depend on the specific way one chooses to do the computation. Any scheme that results in a finite on-shell action for arbitrary solutions of Einstein's equations with negative cosmological constant will have this property.

Motivated by the AdS/CFT correspondence, it was proposed in Ref.[4] to define the stress energy tensor associated with asymptotically AdS spacetimes as the metric variation of the renormalized on-shell action. Explicit formulae for the stress energy tensor in terms of coefficients in the asymptotic expansion of the metric near infinity was obtained in Ref.[5] The stress energy tensor so defined does not transform covariantly under all bulk diffeomorphisms because the on-shell renormalized action transforms anomalously. Notice that the energy momentum tensor may transform anomalously even if the anomaly in the action vanishes for a given background. In other words, the value of the anomaly may vanish, but not the value of its metric variation. This case is encountered for global AdS. Only if the anomaly vanishes identically will the stress energy tensor transform covariantly. This is true for even dimensional asymptotically AdS spaces. In Ref.[2] it was argued that the counterterm subtraction method is not covariant (in odd dimensions). Indeed, this is the case. The non-covariance, however, is forced upon us by infrared divergences in the on-shell action, and these divergences cannot be ignored.

This paper is organized as follows. In the next section we recall the definition of asymptotically AdS spacetimes. In section 3, we review the computation of the infrared divergences and the construction of the on-shell renormalized action. Section 4 contains the computation of the stress energy tensor.

2. Asymptotically AdS Spacetimes

Anti-de Sitter (AdS) spacetime is a maximally symmetric solution of Einstein's equations with negative cosmological constant,

$$R_{\mu\nu} - \frac{1}{2}RG_{\mu\nu} = \Lambda G_{\mu\nu}, \tag{1}$$

where Λ is the negative cosmological constant. AdS spacetime is homogeneous and isotropic and its isometry group is $SO(2,d)$. These conditions imply that the AdS solution is conformally flat and satisfy the stronger equation,

$$R_{\kappa\lambda\mu\nu} = l^2(G_{\kappa\mu}G_{\lambda\nu} - G_{\kappa\nu}G_{\lambda\mu}), \tag{2}$$

where l is related to the cosmological constant as $\Lambda = -d(d-1)/2l^2$.

The metric for AdS_{d+1} is given by

$$ds^2 = l^2 \left[-(1+r^2)dt^2 + \frac{dr^2}{(1+r^2)} + r^2 d\Omega_{d-1} \right]. \tag{3}$$

At infinity the space has a boundary with topology $R \times S^{d-1}$ (we actually consider the covering space of AdS). Let us introduce a new coordinate $\tan\theta = r$. The metric becomes

$$ds^2 = \frac{l^2}{\cos^2\theta} \left[-dt^2 + d\theta^2 + \sin^2\theta d\Omega_{p-1} \right]. \tag{4}$$

The boundary is now located at $\theta = \pi/2$. The metric has a second order pole at the boundary, so it does not induce a metric there. To obtain a metric one picks a positive function r with a single zero at the boundary, and then evaluates $r^2 G$ at the boundary,

$$g_{(0)} = r^2 G|_{R \times S^{d-1}}. \tag{5}$$

In our case, one can pick $r = \cos\theta$. Then $g_{(0)}$ is the standard metric on a Lorentzian cylinder. The metric $g_{(0)}$, however, depends on the choice of defining function. Different defining functions are related by $r' = re^w$. It follows that $g_{(0)}$ is well-defined up to conformal transformation. Thus the AdS metric induces a conformal structure (i.e. a metric up to conformal transformations) at the boundary.

To define asymptotically AdS spacetimes it is useful to recall the notion of conformal infinity introduced by Penrose. Let X be the interior of a manifold-with-boundary $\bar{X}$, M the (compact) boundary of $\bar{X}$, and G a metric on X. The metric G is said to be conformally compact if there exists a defining function r (i.e. $r > 0$ in X and $r = 0, dr \neq 0$ on M) such that $r^2 G$ can be smoothly extended as a metric on $\bar{X}$. This procedure defines a conformal structure at the boundary in the way described in the previous paragraph. If in addition G satisfies (1) we will say that X is an "asymptotically *AdS* spacetime". Notice that that this definition is more general than the one employed in Ref.[2] since we do not impose any restriction on the topology of the boundary.

Given a conformal structure, can one obtain an asymptotically AdS spacetime with this conformal structure at infinity? One may view this question as a Dirichlet problem for AdS gravity. This question has been investigated in the mathematics literature. It was shown by Fefferman and Graham in Ref.[6] that one can obtain an asymptotic solution of Einstein's equations given a representative of the conformal structure. The solution is of the form

$$\begin{aligned} ds^2 = G_{\mu\nu}dx^\mu dx^\nu &= l^2 \left(\frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} g_{ij}(x,\rho) dx^i dx^j \right), \\ g(x,\rho) &= g_{(0)} + \cdots + \rho^{d/2} g_{(d)} + h_{(d)} \rho^{d/2} \log\rho + ... \end{aligned} \tag{6}$$

Several comments are in order here:

(i) Any asymptotically AdS metric can be brought into this form near the boundary. The boundary is located at $\rho = 0$.

(ii) Einstein's equations can be solved order by order in the r variable. Given a metric $g_{(0)}$ one can uniquely determine the coefficients $g_{(2)}, ..., g_{(d-2)}$ and $h_{(d)}$ in terms of $g_{(0)}$.

(iii) The coefficient $h_{(d)}$ is present only when d is even. It is equal to the metric variation of the coefficient of the infrared logarithmic divergence in the on-shell gravitational action (discussed below). It is traceless and covariantly conserved.

(iv) The asymptotic analysis determines only the trace and covariant divergence of $g_{(d)}$.

The explicit expression for $g_{(2)}, ..., g_{(d-2)}, h_{(d)}$ and the equations for the trace and covariant divergence of $g_{(d)}$ can be found in appendix A of Ref.[5]

A case where the Dirichlet problem can be solved exactly is when the bulk metric is conformally flat.[7] In this case one obtains an exact solution given by the metric (6) with

$$\begin{aligned} &g(x,\rho) = g_{(0)}(x) + g_{(2)}(x)\rho + g_{(4)}(x)\rho^2 \quad , \\ &g_{(2)ij} = \frac{1}{d-2}(R_{ij} - \frac{1}{2(d-1)} R g_{(0)ij}) \quad , \quad g_{(4)} = \frac{1}{4}(g_{(2)})^2 \quad , \end{aligned} \tag{7}$$

where the curvatures are of the metric $g_{(0)}$.[a]

So far we have discussed the construction of a bulk metric given a representative of a conformal structure. One may ask how these results change if one picks a different representative of the given conformal class, i.e. how the coefficients $g_{(i)}$ transform if we let $g_{(0)} \to e^{2\sigma(x)} g_{(0)}$. One may use the explicit formulae of $g_{(i)}$ in terms of $g_{(0)}$ in order to obtain these transformation rules. An alternative way to determine them is to note that there are bulk diffeomorphism that preserve the form of the metric (6) and induce the transformation $g_{(0)} \to e^{2\sigma(x)} g_{(0)}$. The infinitesimal form of these bulk diffeomorphisms has been worked out in Ref.[8] Here we discuss the corresponding finite transformations. Consider the coordinate transformation

$$\rho = \rho' e^{-2\sigma(x')} + \sum_{k=2} a_{(k)}(x') \rho'^k, \qquad x^i = x'^i + \sum_{k=1} a^i_{(k)}(x') \rho'^k. \tag{8}$$

The requirement that the transformation leaves the form of the metric invariant uniquely fixes the coefficients $a_{(i)}$ and $a^i_{(i)}$. The first few are given by

$$\begin{aligned} &a_{(2)} = -\frac{1}{2}(\partial\sigma)^2 e^{-4\sigma}, \qquad a_{(3)} = \frac{1}{4} e^{-6\sigma} \left(\frac{3}{4}(\partial\sigma)^2 + \partial^i\sigma\partial^j\sigma g_{(2)ij} \right), \\ &a^i_{(1)} = \frac{1}{2}\partial^i \sigma e^{-2\sigma}, \qquad a^i_{(2)} = -\frac{1}{4} e^{-4\sigma} \left(\partial_k \sigma g^{ik}_{(2)} + \frac{1}{2}\partial^i\sigma(\partial\sigma)^2 + \frac{1}{2}\Gamma^i_{kl}\partial^k\sigma\partial^l\sigma \right), \end{aligned} \tag{9}$$

where indices are raised and lowered with $g_{(0)}$. With these results one can calculate the transformation rules of $g_{(i)}$:

$$g'_{(0)ij} \;\; = \;\; e^{2\sigma} g_{(0)ij}$$

[a] The formula for $g_{(2)}$ is valid for $d \neq 2$. The $d = 2$ case is also covered in Ref.[7]

$$
\begin{aligned}
g'_{(2)ij} &= g_{(2)ij} + \nabla_i\nabla_j\sigma - \nabla_i\sigma\nabla_j\sigma + \frac{1}{2}(\nabla\sigma)^2 g_{(0)ij} \\
g'_{(4)ij} &= e^{-2\sigma}\Big[g_{(4)ij} - 2\sigma h_{(4)ij} - \frac{1}{4}\nabla^k\sigma(\nabla_i g_{(2)jk} + \nabla_j g_{(2)ik} - 2\nabla_k g_{(2)ij}) \\
&+ \frac{1}{4}(\nabla_i\nabla^k\sigma g_{(2)kj} + \nabla_j\nabla^k\sigma g_{(2)ki}) + \frac{1}{4}R_{kilj}\nabla^k\sigma\nabla^l\sigma + \frac{1}{4}\nabla_i\nabla_j\sigma(\nabla\sigma)^2 \\
&+ (\frac{1}{16}(\nabla\sigma)^4 - \frac{1}{4}\nabla^k\sigma\nabla^l\sigma g_{(2)kl})g_{(0)ij} + \frac{1}{4}\nabla_i\nabla_k\sigma\nabla_j\nabla^k\sigma \\
&- \frac{1}{8}(\nabla_i(\nabla\sigma)^2\nabla_j\sigma + \nabla_j(\nabla\sigma)^2\nabla_i\sigma)\Big]. \\
g'_{(d)ij} &= e^{-(d-2)\sigma}g_{(d)ij}, \qquad d = 2k+1. \qquad (10)
\end{aligned}
$$

The infinitesimal version of (9) and (10) agree with the infinitesimal ones derived in Ref.[8].

3. Infrared Divergences

We have argued in the previous section that asymptotically AdS spacetimes come equipped with a bulk Einstein metric and a corresponding boundary conformal structure. We will now show that infrared divergences in the on-shell value of the action force a dependence on the boundary metric, not just on its conformal class.

The gravitational equations (1) can be derived from the action

$$
S[G] = \frac{1}{16\pi G_{\rm N}}[\int_X d^{d+1}x\,\sqrt{G}\,(R[G]+2\Lambda) - \int_{\partial X} d^dx\,\sqrt{\gamma}\,2K], \qquad (11)
$$

where K is the trace of the second fundamental form and γ is the induced metric on the boundary.

The on-shell gravitational action diverges because of the infinite bulk volume and the fact that the induced metric diverges at the boundary. To regulate the theory we cutoff the radial integration, $\rho \geq \epsilon$, and evaluate the boundary term at $\rho = \epsilon$, where ϵ is the regulator. This regularization procedure was proposed in Ref.[9] and implemented in Ref.[3]

Using the asymptotic solution discussed in the previous section one can evaluate the on-shell action. The result is [3]

$$
\begin{aligned}
S_{\rm reg} &= \frac{1}{16\pi G_{\rm N}}\left[\int_{\rho\geq\epsilon} d^{d+1}x\,\sqrt{G}\,(R[G]+2\Lambda) - \int_{\rho=\epsilon} d^dx\sqrt{\gamma}\,2K\right] \\
&= \frac{1}{16\pi G_{\rm N}}\int d^dx\sqrt{\det g_{(0)}}\,\Big(\epsilon^{-d/2}a_{(0)} + \epsilon^{-d/2+1}a_{(2)} + \ldots + \epsilon^{-1}a_{(d-2)} \\
&\qquad - \log\epsilon\, a_{(d)}\Big) + \mathcal{O}(\epsilon^0), \qquad (12)
\end{aligned}
$$

where the coefficients $a_{(n)}$ are local covariant expressions of the metric $g_{(0)}$ and its curvature tensor. The explicit expressions can be found in appendix B of Ref.[5] Here we only give the coefficients of the logarithmic divergences in $d=2$ and $d=4$: $a_{(2)} = \frac{1}{2}R$ and $a_{(4)} = -R^{ij}R_{ij}/8 + R^2/24$.

This computation is universal and applies to any asymptotically AdS spacetime. It is simple to show that the infrared divergences depend only on the coefficients $g_{(2)}, ..., g_{(d-2)}$ in the asymptotic expansion of the metric. As we have seen in the previous section, these coefficients are universal in the sense that they uniquely determined in terms of $g_{(0)}$. The result in (12) does depend on the chosen regularization. The power law divergences are regularization dependent. For instance, a manifestly supersymmetric regularization will yield $a_{(0)} = 0$. The logarithmic divergence, however, is regularization independent.

We now proceed to renormalize the on-shell gravitational action by adding counterterms to cancel the infrared divergences.[3] Using minimal subtraction, we obtain for the renormalized action

$$\begin{aligned} S_{\rm ren}[g_{(0)}] \quad &= \quad \lim_{\epsilon \to 0} \left(S_{\rm reg} - \frac{1}{16\pi G_{\rm N}} \int_{\rho=\epsilon} \sqrt{\gamma}[2(1-d) + \frac{1}{d-2} R \right. \\ &- \left. \frac{1}{(d-4)(d-2)^2}(R_{ij}R^{ij} - \frac{d}{4(d-1)}R^2) - \log\epsilon\, a_{(d)}] \right), \qquad (13) \end{aligned}$$

where we have written the counterterms in terms of fields on the regulating hypersurface $\rho = \epsilon$, as in ref.[4]. This formula should be understood as containing only divergent counterterms in each dimension. This means that in even dimension $d = 2k$ one should include only the first k counterterms and the logarithmic one. In odd $d = 2k+1$, only the first $k+1$ counterterms should be included. The logarithmic counterterms appear only for d even. The renormalized action (13) is finite up to $d = 6$. It is straightforward but tedious to compute the necessary counterterms for $d > 6$. As we have remarked above, only the logarithmic term has an invariant meaning. The other counterterms are regularization and scheme dependent. In particular one can add further boundary terms provided they do not diverge. We will make use of this freedom in the next section. The logarithmic term was incorrectly (and surprisingly) omitted in a large part of the literature on the subject. Because of this counterterm the on-shell action depends on the chosen representative of the boundary conformal structure, i.e.

$$S_{\rm ren}[e^{2\sigma} g_{(0)}] = S_{\rm ren}[g_{(0)}] + \mathcal{A}[g_{(0)}, \sigma]. \qquad (14)$$

The anomalous term has been computed for infinitesimal σ in Ref.[3] It is proportional to $a_{(d)}$. This anomalous transformation has been called the holographic Weyl anomaly because it corresponds to the Weyl anomaly of the corresponding CFT in the AdS/CFT correspondence. Notice that in even dimensions $a_{(2k+1)}$ vanishes identically, so in these cases the on-shell action depends only on the conformal class of the boundary metric. This is in agreement with the fact that conformal field theories in odd dimensions do not have conformal anomalies.

As we have discussed in the previous section, boundary conformal transformations are induced by a specific class of bulk diffeomorphisms. It follows that the finite on-shell action in odd dimensions is *not* invariant under these diffeomorphisms. In other words, infrared divergences break part of bulk diffeomorphisms. Only the

bulk diffeomorphisms that do not yield a boundary Weyl transformation are true symmetries. The anomaly itself is a conformal invariant, so its vanishing depends only on the conformal class of the boundary metric. It may vanish in one background, but its metric variation may not. This means that in backgrounds where the anomaly vanishes non-trivially, so the on-shell action is invariant under all diffeomorphisms, the stress energy tensor may still have an anomalous variation. This is exactly what happens for (globally) AdS spacetimes.

4. Stress Energy Tensor

In the previous section we have obtained the on-shell value of the gravitational action as a functional of the boundary metric. In Ref.[10], Brown and York proposed defining the gravitational energy momentum tensor through a Hamilton-Jacobi analysis as the functional derivative of the on-shell action with respect to the boundary metric. In Ref.[4] Balasubramanian and Kraus considered the Brown-York energy momentum tensor for asymptotically AdS spaces. Including the counterterms of Ref.[3] (except for the logarithmic one), they showed that the resulting energy momentum tensor was finite in the cases they considered. Their analysis was extended in Ref.[5], where completely explicit expressions for the energy momentum tensor in any even dimension and up seven dimensions were obtained, as we now review.

The stress energy tensor is defined to be

$$T_{ij} = \frac{2}{\sqrt{\det g_{(0)}}} \frac{\delta S_{\rm ren}}{\delta g^{ij}_{(0)}}. \tag{15}$$

This can be evaluated by first computing the energy momentum tensor in the regulated theory and then removing the regulator,

$$T_{ij} = \lim_{\epsilon\to 0} \frac{2}{\sqrt{\det g(x,\epsilon)}} \frac{\delta S_{\rm ren}}{\delta g^{ij}(x,\epsilon)} = \lim_{\epsilon\to 0} \left(\frac{1}{\epsilon^{d/2-1}} T_{ij}[\gamma] \right), \tag{16}$$

where $T_{ij}[\gamma]$ is the stress energy tensor in terms of the induced metric γ of the hypersurface $\rho = \epsilon$. It receives a contribution $T^{\rm reg}[\gamma]$ from the original (regulated) action (12), and a contribution $T^{\rm ct}[\gamma]$ from the counterterms. The former is equal to $T^{\rm reg}[\gamma] = -\frac{1}{8\pi G_{\rm N}}(K_{ij} - K\gamma_{ij})$; the latter can be easily calculated from the explicit form of the counterterms.

Equipped with the explicit asymptotic solutions one can now evaluate (16). This is a rather tedious exercise. The details can be found in Ref.[5]. Here we only give the final result,

$$T_{ij} = \frac{dl}{16\pi G_{\rm N}} (g_{(d)ij} + X^{(d)}_{ij}), \tag{17}$$

where $X^{(d)}_{ij}$ depends on the dimension. The result for all odd d, and even d up to six are:

$$X^{(2k+1)}_{ij} = 0, \qquad X^{(2)}_{ij} = -g_{(0)ij} \,{\rm Tr}\, g_{(2)}$$

$$
\begin{aligned}
X^{(4)}_{ij} &= -\frac{1}{8}g_{(0)ij}[(\mathrm{Tr}\ g_{(2)})^2 - \mathrm{Tr}\ g^2_{(2)}] - \frac{1}{2}(g^2_{(2)})_{ij} + \frac{1}{4}g_{(2)ij}\mathrm{Tr}\ g_{(2)} \\
X^{(6)}_{ij} &= -A_{(6)ij} + S_{ij}
\end{aligned} \tag{18}
$$

where

$$
\begin{aligned}
A_{(6)ij} = \frac{1}{3}\Bigg(& 2(g_{(2)}g_{(4)})_{ij} + (g_{(4)}g_{(2)})_{ij} - (g^3_{(2)})_{ij} + \frac{1}{8}\left[\mathrm{Tr}\ g^2_{(2)} - (\mathrm{Tr}\ g_{(2)})^2\right] g_{(2)ij} \\
& -\mathrm{Tr}\ g_{(2)}\left[g_{(4)ij} - \frac{1}{2}(g^2_{(2)})_{ij}\right] - \left[\frac{1}{8}\mathrm{Tr}\ g^2_{(2)}\mathrm{Tr}\ g_{(2)} - \frac{1}{24}(\mathrm{Tr}\ g_{(2)})^3\right. \\
& \left. -\frac{1}{6}\mathrm{Tr}\ g^3_{(2)} + \frac{1}{2}\mathrm{Tr}\ (g_{(2)}g_{(4)})\right] g_{(0)ij}\Bigg)
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
S_{ij} = \frac{1}{24}\Bigg(& \nabla^2 C_{ij} - 2R^{k\ l}_{\ i\ j}C_{kl} + 4(g_{(2)}g_{(4)} - g_{(4)}g_{(2)})_{ij} + \frac{1}{10}(\nabla_i\nabla_j B - g_{(0)ij}\nabla^2 B) \\
& + \frac{2}{5}g_{(2)ij}B + g_{(0)ij}\Big(-\frac{2}{3}\mathrm{Tr}\ g^3_{(2)} - \frac{4}{15}(\mathrm{Tr}\ g_{(2)})^3 + \frac{3}{5}\mathrm{Tr}\ g_{(2)}\mathrm{Tr}\ g^2_{(2)}\Big)\Bigg),
\end{aligned}
$$

$$
C_{ij} = (g_{(4)} - \frac{1}{2}g^2_{(2)} + \frac{1}{4}g_{(2)}\mathrm{Tr}\ g_{(2)})_{ij} + \frac{1}{8}g_{(0)ij}B \quad , \quad B = \mathrm{Tr}\ g^2_2 - (\mathrm{Tr}\ g_2)^2 \quad .
$$

The vanishing of $X^{(2k+1)}_{ij}$ reflects the fact that the holographic Weyl anomaly identically vanishes in even spacetimes. In the computation we used the freedom to add finite counterterms in order to to remove terms proportional to $h_{(d)}$. We remind the reader that $h_{(d)}$ is proportional to the metric variation of $a_{(d)}$ so by adding a finite counterterm proportional to $a_{(d)}$ one can remove all $h_{(d)}$ dependence from the energy momentum tensor.

One can check by explicit computation that the stress energy tensors so obtained are covariantly conserved and their trace correctly reproduces the holographic Weyl anomaly.

The stress energy transforms anomalously under bulk diffeomorphisms that induce Weyl transformations in the boundary. Indeed, using the definition (15) and the transformation of the on-shell action one gets,

$$
T_{ij}[e^{2\sigma}g_{(0)}] = e^{-(d-2)\sigma}\left(T_{ij}[g_{(0)}] + \frac{1}{\sqrt{g_{(0)}}}\frac{\delta\mathcal{A}[g_{(0)}]}{\delta g^{ij}_{(0)}}\right). \tag{20}
$$

From here it follows that the stress energy tensor may transform anomalously even if $\mathcal{A}$ evaluated on a particular background is equal to zero. An alternative way to compute the dependence of the stress energy tensor on a given representative of the boundary conformal structure is to use the transformation rules of $g_{(i)}$ in (10) and formula (17)-(19). The infinitesimal transformations can be found in Ref. [5].

To summarize: given a solution of Einstein's equation with negative cosmological constant one can obtain the corresponding energy momentum tensor by first reaching the coordinate system (6) near the boundary, and then plugging in the coefficients $g_{(i)}$ in (17).

We finish this section with a few comments on the addition of matter fields. As in the case of pure gravity one first needs to obtain asymptotic solutions of the coupled gravity-matter system. It can happen that the leading behavior of the matter stress energy tensor is more singular than the leading behavior of the Einstein tensor. In this case the methods discussed here apply only if the fields parameterizing the matter boundary conditions are considered infinitesimal. In the other cases one can proceed as in the case of pure gravity to regularize, renormalize the theory and obtain the stress energy tensor as the metric variation of the on-shell action. A more detailed discussion can be found in Ref.[5]

4.1. *Example: AdS$_5$*

As an example we discuss in some detail the case of AdS$_5$ spacetime. The coordinate transformation that brings the metric (3) to the coordinate system in (6) is

$$r^2 = \frac{1}{\rho}(1 - \frac{\rho}{4})^2. \tag{21}$$

The metric coefficients are given by $g_{(0)} = \mathrm{diag}(-1, 1, \sin^2\theta, \sin^2\theta \sin^2\phi)$, $g_{(2)} = -\frac{1}{2}\mathrm{diag}(1, 1, \sin^2\theta, \sin^2\theta \sin^2\phi)$, and $g_{(4)} = \frac{1}{16} g_{(0)}$. The expansion terminates at ρ^2 because AdS$_5$ is conformally flat.[7] The expressions for $g_{(2)}$ and $g_{(4)}$ agree with formula (7). Using (17) we now get

$$T_{ij} = \frac{l^3}{64\pi G_{\rm N}}(4\delta_{i,0}\delta_{j,0} + g_{(0)ij}) \tag{22}$$

One can explicitly check that this stress energy tensor is conserved and traceless. It is traceless because the conformal anomaly evaluated for global AdS vanishes.

The boundary metric is conformally flat. The coordinate transformation

$$\tau \pm r = \tan\frac{1}{2}(t \pm \theta) \tag{23}$$

brings the metric to the form

$$ds^2_{(0)} = 4\cos^2\frac{1}{2}(t+\theta)\cos^2\frac{1}{2}(t-\theta)\left(-d\tau^2 + dr^2 + r^2 d\Omega_2\right) \tag{24}$$

We can implement this Weyl transformation using the bulk diffeomorphism (8) with $e^{-2\sigma} = 4\cos^2\frac{1}{2}(t+\theta)\cos^2\frac{1}{2}(t-\theta)$. One can obtain the new metric by working out (10). After some algebra one obtains $g'_{(2)} = g'_{(4)} = 0$. Alternatively, one can notice that the Dirichlet boundary problem for conformally flat bulk metrics has a unique answer. In our case, after the transformation $g'_{(0)}$ is flat, and from (7) we get $g'_{(2)} = g'_{(4)} = 0$. It follows that T_{ij} vanishes identically.

We thus see explicitly that the energy momentum tensor, defined as the metric variation of the renormalized on-shell action, is *not* covariant with respect to diffeomorphisms that yield a Weyl transformation in the boundary. The anomalous transformation has its origin in the infrared divergences in the computation of the on-shell action.

Notice that these results are in exact agreement with expectations from the AdS/CFT correspondence. The gravitational energy momentum tensor is identified with the expectation value of the CFT stress energy tensor. This expectation value is zero for a CFT on flat space. When we consider the CFT on $R \times S^3$, however, the expectation value is non-zero due to the Casimir energy. Using the AdS/CFT dictionary for the gravity/gauge theory parameters one finds exact agreement [4]. One could infer this agreement from the fact that the Casimir energy follows from the trace anomaly, and the latter was shown to be reproduced exactly by a gravity computation in Ref.[3]. These quantities had to agree, even though the gravity computation is at strong coupling and the CFT one in weak coupling, because there is a non-renormalization theorem that protects them.

To shed some more light on the results for the stress energy tensor, one can work out the relation between the generator $H = \partial/\partial t$ of global time translations to generators of isometries of AdS with flat boundary. Using the coordinate transformation (23), one obtains

$$H = \frac{1}{2}(P_\tau + K_\tau), \tag{25}$$

where $P_\tau = \partial/\partial\tau$ is the generator of τ-time translations and $K_i = x^2\partial_i - 2x_i x^i \partial_j$ is the generator of special conformal transformations. In a theory with conformal anomalies, the dilatations and special conformal transformations are broken. This allows for a non-zero eigenvalue of the generator of global time translations acting on the ground state. Notice that in the case of Euclidean AdS global time translations are mapped to dilatations, which are also broken.

Acknowledgements

This research is supported in part by the NSF grant PHY-9802484.

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